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# A class of solutions for steady stratified flows

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Under the assumption that the horizontal scales of the flow of a stratified fluid are much greater than the vertical scale, it can be shown that the pressure distribution in the fluid is nearly hydrostatic and that the solution for steady flows can be reduced to the solution of a non-linear partial differential equation with only horizontal co-ordinates as the space variables. The theory built on the basic assumption is the shallow-water theory for stratified fluids. Transformations are explicitly given with which a class of solutions for steady three-dimensional flows of a fluid of arbitrary stratification, continuous or discontinuous, issuing from a large reservoir can be found from a corresponding solution for a homogeneous fluid, provided a free surface is present and the shallow-water theory is applicable. A few examples of exact solutions according to the shallow-water theory are given and the parallel flow in a horizontal canal issuing from a large reservoir with the same horizontal bottom, which has some bearing on previous works on stratified flows, is discussed. But it is emphasized that the class is a very special one and that there are other solutions not belonging to this class. The conditions under which a solution belonging to this class is valid are discussed.

## 1. Introduction

The equations governing large-amplitude three-dimensional steady flows of a stratified fluid have been presented in a previous paper (Yih 1967). Due to the non-linearity and complexity of these equations not a single solution for a truly three-dimensional case is known. If, however, the vertical scale of a stratified liquid is small compared with a representative horizontal scale, the pressure distribution at any section is essentially hydrostatic. As a consequence the number of the spatial variables can be reduced from three to two, although the flow treated is still truly three-dimensional. The theory built on the basic assumption of small vertical scale is the so-called shallow-water theory. In this paper we shall show that, whenever the shallow-water theory assumption is valid, a class of exact solutions exists for steady flows of a stratified fluid. The principal result is that to any solution by the shallow-water theory for a steady flow of a homogeneous fluid there corresponds a solution for a steady flow of a stratified fluid with arbitrary stratification, the velocity field for the latter being obtained from that for the former by a transformation explicitly dependent on the density stratification, and that steady stratified flows issuing from a large reservoir enjoy this correspondence, provided the assumptions underlying the shallow-water theory are satisfied and the downstream conditions allow it.

## 2. The basic assumptions and their principal consequence

We restrict our attention to the case of an incompressible, inviscid and non-diffusive fluid of variable density. For such a fluid the equations of motion are

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x}, \tag{1}$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y}, \tag{2}$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - g\rho. \tag{3}$$

In these equations $x$, $y$ and $z$ are Cartesian co-ordinates; $u$, $v$ and $w$ are velocity components in the directions of increasing $x$, $y$ and $z$, respectively; $p$ is the pressure, $\rho$ the density, $g$ the gravitational acceleration, which is in the direction of decreasing $z$, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

The equation of continuity is

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

and the equation of incompressibility is

$$D\rho/Dt = 0, \tag{4}$$

in virtue of which the equation of continuity can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{5}$$

We shall consider flows of which $u$ and $v$ are of the order of a representative velocity $V_0$, and the representative horizontal length is $L$. Furthermore, for unsteady flows, if

$$f = O(E) \quad \text{and} \quad \partial f/\partial t = O(\omega E),$$

in which $f$ as well as $E$ is any variable, we shall say that $\partial/\partial t$ is of the order of $\omega$, for convenience.

If we denote by $\zeta$ the vertical displacement of a fluid particle from its upstream elevation or mean elevation, then

$$\frac{\partial \zeta}{\partial t} + u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = w. \tag{6}$$

The basic assumption of the shallow-water theory is

$$\text{(i)} \quad (h/L)^2 = \epsilon^2 \ll 1,$$

in which $h$ is the depth of the fluid and a function of $x$, $y$ and $t$. Since $\zeta < h$, (i) implies that $(\zeta/L)^2 < \epsilon^2 \ll 1$. We shall assume that the bottom is flat and situated at $z = 0$. Integration of (5) with respect to $z$ then produces the result

$$w = O(\epsilon V_0), \tag{7}$$

in which $V_0$ is a representative velocity. Now the second and third terms in (6) are also $O(\epsilon V_0)$; hence

$$\partial\zeta/\partial t = O(\epsilon V_0) \quad \text{and} \quad \partial/\partial t = O(V_0/L).$$

Substituting (7) into (3) and ignoring quantities of order $\epsilon$ or of higher order in $\epsilon$, we then obtain

$$\frac{\partial p}{\partial z} = -g\rho \quad \text{or} \quad p = \int_z^h g\rho\, dz, \tag{8}$$

if the free surface is present and given by

$$z = h(x, y, t). \tag{9}$$

In this paper we assume that a free surface is present. Equation (8) is the principal consequence of (i). Subsequent developments will be for steady flows only.

## 3. Shallow-water theory for stratified liquids in steady flow

For the development of the shallow-water theory for stratified fluids in steady flow, a presentation of the shallow-water theory for a homogeneous liquid is essential. Consider a homogeneous fluid with a free surface flowing above a horizontal bed. The depth will be denoted by $h$. If we assume the upstream flow to be irrotational, or, more generally, the flow to have been started from rest, then the whole flow is irrotational, since the fluid is inviscid and the density constant. Since (7) is still valid under the basic assumption (i), the equations of irrotationality are, if $U$ and $V$ denote $u$ and $v$ for homogeneous fluids,

$$\frac{\partial V}{\partial z} = 0, \quad \frac{\partial U}{\partial z} = 0, \quad \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} = 0, \tag{10}$$

if quantities of order $\epsilon$ are neglected. The first two equations in (10) state simply that $U$ and $V$ are independent of $z$ and the third allows the use of a velocity potential $\Phi$ in terms of which

$$U = \partial\Phi/\partial x, \quad V = \partial\Phi/\partial y. \tag{11}$$

The equation of continuity is then, as can be shown in the usual way by taking as the control surface the surface formed by the bottom, the free surface and the lateral surface of a vertical prism of cross-section $dx\,dy$,

$$\frac{\partial(Uh)}{\partial x} + \frac{\partial(Vh)}{\partial y} = 0. \tag{12}$$

The Bernoulli equation is, with $h_0$ denoting the depth at a stagnation point or in a large reservoir,

$$U^2 + V^2 + 2gh = 2gh_0, \tag{13}$$

since the neglected term $w^2$ is of the order of $\epsilon^2$. Instead of the equations of motion, we can simply use (13), which can be derived from them. Substitution of (13) into (12) produces

$$(c^2-U^2)\,U_x - UV(U_y+V_x) + (c^2-V^2)\,V_y = 0, \tag{14}$$

in which

$$c^2 = gh. \tag{15}$$

In virtue of (11), (14) can be written as

$$(c^2-\Phi_x^2)\,\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + (c^2-\Phi_y^2)\,\Phi_{yy} = 0, \tag{16}$$

in which subscripts indicate partial differentiation. The $c^2$ in (14) and (16) can be expressed in terms of $U$ and $V$ by the use of (13), or in terms of $\Phi$ in the further use of (11). It was Riabouchinsky (1932) who first pointed out the analogy between (16) and the equation governing the velocity potential of two-dimensional irrotational flows of a homentropic inviscid gas. Equation (16) is, as Riabouchinsky pointed out, identical to the equation governing two-dimensional irrotational motion of a homentropic gas obeying the law for isentropic change of state

$$p/\rho^\gamma = \text{constant}, \quad \text{with} \quad \gamma = 2.$$

The equation in gas dynamics corresponding to (16) has been studied by Molenbroek (1890) and more fruitfully by Chaplygin (1904), both of whom used hodographic variables as independent variables.

Now, for the motion of a stratified liquid started from rest, vorticity will be created. But the vortex lines will lie in surfaces of constant density, so that the vorticity component normal to a surface of constant density is zero, as a direct consequence of the Kelvin theorem (see Yih 1965, pp. 13–14). Remembering the assumption (i), this means that, with terms of order $\epsilon$ neglected, the third equation in (10) still stands. Thus we have irrotationality in a constant-density surface, when the motion is viewed from above. This does not save (11) for the whole field of flow, but does save it for a constant-density surface. As to the first two equations in (10), they are certainly no longer valid.

We shall now show that, if the shallow-water assumption is satisfied, steady flow of a stratified fluid with a free surface issuing from a large reservoir can, although it does not necessarily, have a flow pattern exactly like that of a homogeneous fluid with a free surface, issuing from the same reservoir into the same channel. Since a flow having such a pattern is far from the only kind of flow a stratified fluid can have, it is sufficient to show that such a flow is dynamically permissible, i.e. it is consistent with the only two equations governing the flow: the equation of continuity and the Bernoulli equation.

We shall, then, assume that for every constant-density surface

$$\zeta/h = \zeta_0/h_0, \tag{17}$$

in which $h_0$ is the depth far upstream (in the reservoir), $\zeta_0$ is the reservoir elevation of the constant-density surface, which has the elevation $\zeta(x,y)$ at other places, and $h(x,y)$ the depth at any $(x,y)$. The pressure at any point is, under the shallow-water assumption,

$$p = g\int_z^h \rho(z')\,dz'. \tag{18}$$

At any *fixed* values of $x$ and $y$, $z'$ can be identified with $\zeta$ and $dz'$ with $d\zeta$, and a change in the value in $\zeta$ involves a change in the value of $\rho$. Hence (18) can be written as

$$p = g\int_z^h \rho(\zeta)\,d\zeta = \frac{gh}{h_0}\int_{z_0}^{h_0} \rho_0(\zeta_0)\,d\zeta_0 = \frac{gh}{h_0}\int_{z_0}^{h_0} \rho_0(z_0')\,dz_0', \tag{19}$$

in which the validity of the second equality sign depends on (17).

Whether or not (17) is assumed, the Bernoulli equation written for any point $(x, y, z)$ and a point far upstream on the same constant-density surface is

$$u^2 + v^2 + 2g\left(z + \frac{1}{\rho(z)}\int_z^h \rho(z')\,dz'\right) = C(\rho), \tag{20}$$

in which
$$C(\rho) = 2g\left(z_0 + \frac{1}{\rho_0(z_0)}\int_{z_0}^{h_0} \rho_0(z_0')\,dz_0'\right). \tag{21}$$

If (17) is assumed, (20) and (21) permit (22) to be written as

$$u^2 + v^2 + 2ghB(\rho) = 2gh_0B(\rho), \tag{22}$$

in which
$$B(\rho) = C(\rho)/2gh_0. \tag{23}$$

If we now write
$$(u, v) = \lambda(\rho)\,(U, V), \quad \lambda^2 = B(\rho), \tag{24}$$

(22) becomes
$$U^2 + V^2 + 2gh = 2gh_0, \tag{25}$$

the same as (13). That is, if the Bernoulli equation is satisfied by the flow of a homogeneous fluid, it is also satisfied by a stratified flow with the same flow pattern and a velocity distribution given by (24).

Note that, for steady flows, (6) gives

$$W = U\frac{\partial \zeta}{\partial x} + V\frac{\partial \zeta}{\partial y} \tag{26}$$

for a homogeneous fluid and
$$w = u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} \tag{27}$$

for a stratified fluid. Hence (24) also implies

$$w = \lambda(\rho)\,W. \tag{28}$$

Then, in virtue of (4), (5) is satisfied if

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0 \tag{29}$$

is satisfied, provided the velocity distribution in the stratified fluid is given by (24). The consistency of (17) and (24) with the equation of continuity and with the dynamical equations is thus established, and we state the

*Theorem: So long as the shallow-water theory is valid, a class of steady stratified flows with a free surface originated from rest can be found corresponding to each irrotational steady free-surface flows of a homogeneous fluid originated from rest. The mapping is by the use of* (24).

Note that even in the presence of a stagnant layer of fluid the *flowing part* of the stream can still obey the theorem. In other words, the theorem is true wherever the basic assumptions of the shallow-water theory are fulfilled and a free surface or stagnant upper layer is present. We shall now present a few examples. Examples 3 and 4 illustrate flows with a stagnant layer.

*Example 1. Gravity jets of a stratified fluid*

With reference to figure 1, the water level ($A$) behind the vertical wall far from the opening is higher than the water level in front of the vertical wall, which is flat except in the jet issuing from the opening. The curved free surface of the jet

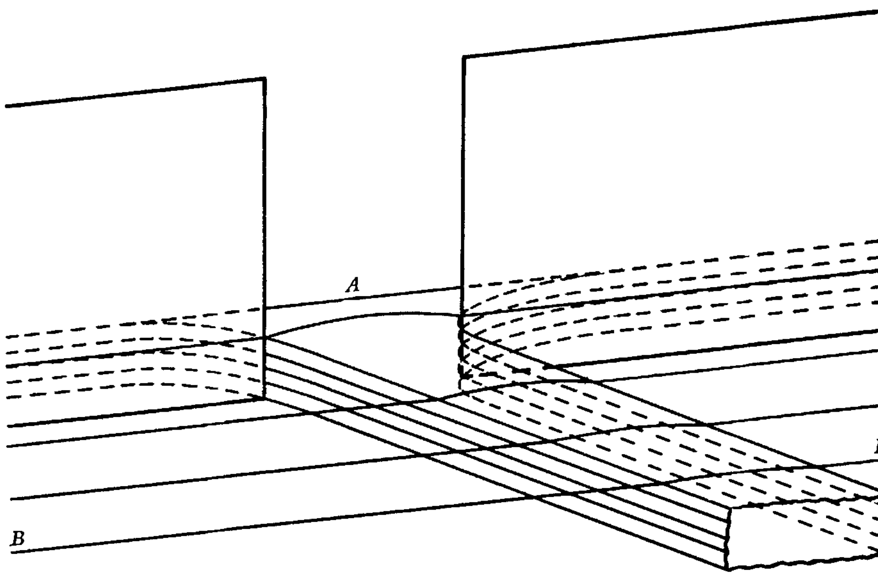


FIGURE 1. A perspective view of the gravity jet. The line $A$ indicates the level of the free surface far upstream. The flat part of the intersection $B$–$B$ of the free surface with a vertical plane is on the surface of the stagnant liquid surrounding the jet. There are three other lines like $B$–$B$ in the figure. Their flat parts are at the same level as the flat part of $B$–$B$. The fluid may be homogeneous or stratified.

is higher than the flat level (straight part of $B$–$B$) of the dead water surrounding it, but approaches that level very far downstream. The bottom is horizontal throughout and the depth of water is assumed to be small compared with the opening in the wall. The problem for a homentropic gas was solved by Chaplygin (1904), and Ferguson & Lighthill (1947) calculated the coefficient of contraction for $\gamma = 1{\cdot}4$. In the shallow-water case $\gamma = 2$, whereas for the classical Kirchhoff jet $\gamma = \infty$. The coefficient of contraction $C_c$ is the ratio of the asymptotic width of the jet to the opening of the wall. For the Kirchhoff jet

$$C_c = \frac{\pi}{\pi + 2} = 0{\cdot}611.$$

For the Chaplygin jet Ferguson & Lighthill (1947) gave $C_c$ for $\gamma = 1{\cdot}4$ and various values of

$$\tau_1 = \frac{q^2}{q_{\max}^2}, \tag{30}$$

$q$ being the speed along the boundary of the jet and $q_{max}$ the maximum speed attainable by the gas. In our case $\gamma = 2$ and $q_{max}$ is now the maximum speed attainable by the liquid.

A calculation by Mr C. H. Li gives, for $\gamma = 2$,

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\tau_1$ | 0 | 0·02 | 0·04 | 0·06 | 0·08 | 0·10 | 0·12 | 0·14 | — |
| $C_c$ | $\pi/(\pi+2)$ | 0·6156 | 0·6205 | 0·6255 | 0·6307 | 0·6362 | 0·6419 | 0·6479 | — |
| $\tau_1$ | 0·16 | 0·20 | 0·22 | 0·24 | 0·26 | 0·28 | 0·30 | 0·32 | $\frac{1}{3}$ |
| $C_c$ | 0·6542 | 0·6677 | 0·6749 | 0·6825 | 0·6904 | 0·6987 | 0·7075 | 0·7167 | 0·7230 |

The maximum value of $\tau_1$ for subcritical flow is $(\gamma-1)/(\gamma+1) = \frac{1}{3}$ for $\gamma = 2$.

For a stratified fluid with any stratification, we need (24) to obtain the velocity distribution. But the coefficient of contraction is the same if the flow pattern remains unchanged. It is tacitly assumed that, if the density far upstream is given by $\rho = f(z)$, that in the stagnant liquid surrounding the jet is given by $\rho = f(rz)$, with $r$ equal to the ratio of the upstream depth to the depth far downstream, if the flow pattern is to remain the same as for a homogeneous liquid. This can be achieved by having two large basins divided by the wall, filling them while keeping the sluice gate open, then closing the gate and enlarging in any way the area of the downstream basin, thus lowering the levels of the constant-density surfaces proportionally. When the gate is then opened, the condition at the edge of the jet is just what is needed for the solution to be physically relevant.

*Example 2. Stratified flow in a channel expansion*

Figure 2 $(a, b)$ shows the plan and elevation (at the centre plane) views of a homogeneous liquid flowing through a channel supercritically, i.e. with the velocity everywhere greater than the local speed of long waves of the gravest mode. Equation (16) is now entirely hyperbolic and the solution by the use of the method of characteristics is well known. For a stratified liquid with any stratification, again (24) provided the corresponding solution.

*Example 3. Gravity jets with an overlying stagnant layer*

In figure 3, if the flowing layer is homogeneous and has the constant density $\rho_l$, the gravity jet will be identical to the gravity jet without an overlying layer in every respect, except that the velocity is reduced by the factor $(\rho_l-\rho')/\rho_l$, $\rho'$ being the density of the overlying layer. This can be easily seen, since the Bernoulli equation is now

$$U'^2+V'^2+2g'h = 2g'h_0, \tag{31}$$

in which

$$g' = \frac{\rho_l-\rho'}{\rho_l}g, \tag{32}$$

and the primes on $U$ and $V$ are to indicate the presence of the overlying layer, for the sake of distinction.

If the flowing layer is stratified, the velocity distribution in a dynamically possible flow with the same flow pattern is given by

$$u = \lambda'(\rho)\,U', \quad v = \lambda'(\rho)\,V', \tag{33}$$

in which $\rho$ now varies from one surface to another and the prime on $\lambda$ does not

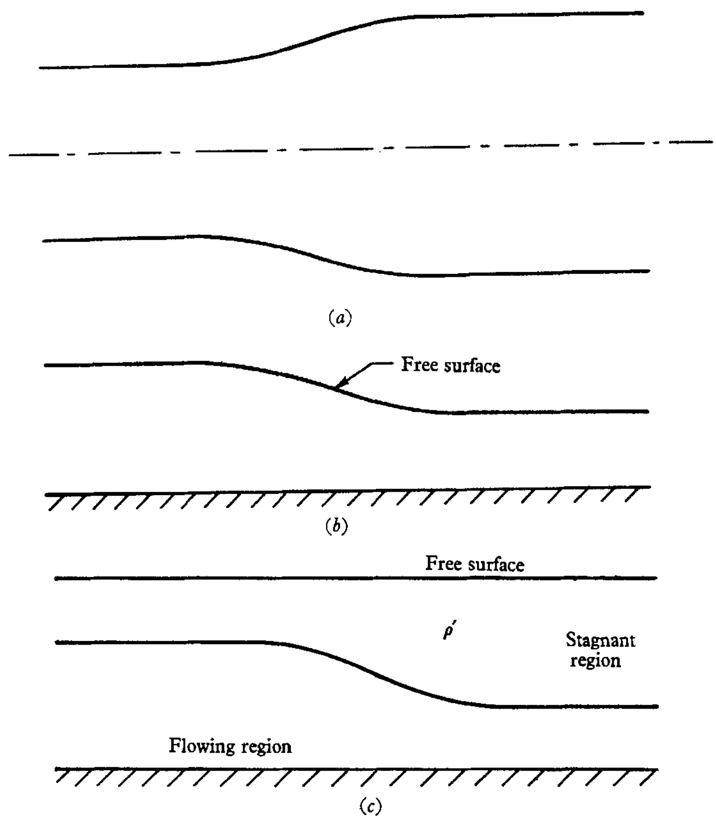


FIGURE 2. (*a*) The plan view of a channel contraction. (*b*) The elevation view of the cross-section along the centre plane. The fluid may be homogeneous or stratified. (*c*) The elevation view of the same cross-section, with an overlying stagnant layer present. The flowing fluid may be homogeneous or stratified.

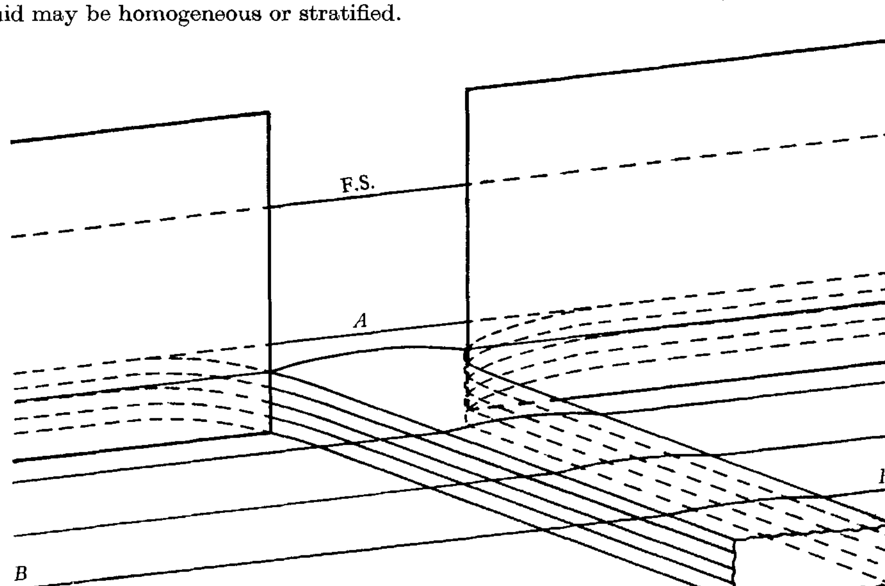


FIGURE 3. A perspective view of the gravity jet with an overlying stagnant layer. Line *A* again marks the elevation of the flowing fluid far upstream. The fluid may be homogeneous or stratified. F.S. is the free surface.

indicate differentiation. It is important to note that $\lambda'(\rho)$ is not proportional to $\lambda(\rho)$ by the factor

$$(\rho_t-\rho')/\rho_t,$$

where $\rho_t$ is now the density at the top stream surface of the flowing liquid. Indeed, we have to determine $\lambda'(\rho)$ anew. To this end we need to write the Bernoulli equation for the stratified fluid, which is

$$u^2+v^2+2g\left[z+\frac{1}{\rho}\int_z^h \rho(z')\,dz' - \frac{\rho'}{\rho}h\right] = C(\rho), \tag{34}$$

if $h$ is again the depth of the flowing layer, $\rho$ the density in a constant-density surface and, from the upstream conditions,

$$C(\rho) = 2gh_0 B'(\rho),$$

in which

$$B'(\rho) = \left[\frac{z_0}{h_0} + \int_{z_0}^{h_0} \frac{\rho_0}{\rho_0(z_0)}\frac{dz_0}{h_0} - \frac{\rho'}{\rho_0(z_0)}\right]. \tag{35}$$

Note that $\rho_0(z_0) = \rho(z)$. Again it can be shown that the assumption is dynamically permissible and that, with it, the Bernoulli equation becomes

$$u^2+v^2+2ghB'(\rho) = 2gh_0B'(\rho), \tag{36}$$

or

$$u^2+v^2+2g'h\frac{\rho_t}{\rho_t-\rho'}B'(\rho) = 2g'h_0\frac{\rho_t}{\rho_t-\rho'}B'(\rho). \tag{37}$$

Thus

$$[\lambda'(\rho)]^2 = \frac{\rho_t}{\rho_t-\rho'}B'(\rho). \tag{38}$$

Since $B'(\rho)$ is not a constant multiple of $B(\rho)$ given by (23) and (21), $\lambda'(\rho)$ is not proportional to $\lambda(\rho)$ given by (24). With $\lambda'(\rho)$ given by (38), (33) gives $u$ and $v$, with

$$(U', V') = \frac{\rho_t-\rho'}{\rho_t}(U, V). \tag{39}$$

Of course we do not have to use the flow $(U', V')$ parametrically and could have related $u$ and $v$ to $U$ and $V$ by inspection of (36). We have used the flow $(U', V')$ chiefly to show more clearly that it is dangerous to apply (24) indiscriminately. Such an indiscriminate application would have given the wrong results

$$u = \lambda(\rho)\,U', \quad v = \lambda(\rho)\,V',$$

with $\lambda(\rho)$ given by (24). Again, it is tacitly assumed that the density far upstream in the flowing layer is obtainable from that in the stagnant liquid surrounding the jet by a stretching of the vertical length scale. This situation can be achieved as explained under example 1, although the overlying fluid must be made level throughout by addition to the lower basin.

*Example 4. Flow through a channel expansion, with an overlying stagnant layer*

Figure 2 $(a, c)$ shows the plan and elevation (through the centre plane) views of a homogeneous liquid of density $\rho_t$ flowing through a channel. The density of the stagnant layer is again $\rho'$. The flow is supposed to be supercritical in the sense that the speed $q$ is everywhere greater than $(g'h)^{\frac{1}{2}}$, with $g'$ given by (32). Again the velocity distribution is given by (33) and (38).

*Example 5. Flow from a reservoir into a channel*

In a previous paper (Yih 1958) it was shown that, if a stratified liquid flows horizontally from an infinitely large reservoir into an adjoining channel with the same horizontal bottom and the same horizontal cover, the velocity distribution in the channel, where the velocity becomes unidirectional, is given by

$$\sqrt{\rho}\, u = \text{constant}. \tag{40}$$

For convenience of reference, we shall call (40) solution A, which is an inertial solution. In other words, it is true only if the acceleration is achieved by very low pressure downstream and gravity plays no role. That is to say, if we define the local Froude number as

$$V_0/\sqrt{(g'h_0)},$$

in which

$$g' = \frac{gh_0}{\rho}\frac{\partial \rho}{\partial z},$$

then the higher the minimum of the Froude number, the more nearly is the velocity distribution given by (40). At low Froude numbers (40), though dynamically possible, is not likely to describe what actually happens, since the flow is then strongly affected by gravity. With a free surface, acceleration is caused by descent of the fluid, and the velocity distribution in a channel joining a reservoir from which the fluid issues is determined by (24), provided the downstream conditions allow such a flow. There is no contradiction of the two results. It matters a great deal whether there is a free surface, and when there is no free surface it matters a great deal how high the Froude number is.

If the upper surface is free, the velocity distribution of a homogeneous fluid issuing *subcritically* from the reservoir into the channel will be free from waves, since no real characteristics exist for (16), according to the shallow-water theory. The velocity distribution far downstream from the contraction will then be uniform. That is, $U$ will be constant. For a stratified fluid, downstream conditions allowing, the asymptotic velocity distribution is given by (24). For convenience we shall call this velocity distribution solution B. We know also that it is possible to have a flowing layer of a homogeneous liquid under a stagnant layer of lighter density. If we use (33) and (38) to determine $u$, with $U' =$ constant, we obtain a flow of a stratified fluid, with an upper part of it stagnant from the reservoir into the channel. Since the upper layer is stagnant, it indeed does not matter whether the upper surface is covered or free. This solution, called solution C, is different from solutions A and B, even granted the same upstream density distribution. But solution C is valid only if blocking has occurred, due to some obstacle downstream. The theorem is still true for the flowing region.

In concluding this section, we remark that there is a weakness in the gravity-jet examples. For the Kirchhoff jets the radius of curvature of the free streamline at the starting-point (as it leaves the wall) is zero. The same is also true for the case of $\gamma = 2$. Professor M. J. Lighthill suggested to the writer in London that the sharp corner at the corresponding point in the hodograph plane guarantees that the curvature at the point in question is infinite. And this turns out to be generally true. Whereas an infinite curvature is no weakness in the Kirchhoff and Chaplygin jets, it is a weakness for the gravity jets discussed here, for at a point with infinite curvature the shallow-water assumption is violated.

In all examples the flow is supposed to have originated from a large reservoir, where the fluid is at rest.

## 4. Discussion

The flows discussed so far belong to a special class. Because of their very special nature it is possible to go far in the description of their detailed features. But the downstream conditions must be consistent with any particular flow belonging to this special class before it can be expected to occur, as we have described, for instance, in example 1. It is certainly desirable to discuss the situation that will prevail for a given density distribution in the upstream reservoir and one in the downstream reservoir connected with the upstream reservoir by an open channel. This will serve to show that other flows than the class just discussed can occur.

If the free surface in the downstream reservoir is sufficiently lower than the free surface upstream, and the velocity determined by (24) is so fast that no internal waves, even of finite amplitude, can travel upstream, then the flows described by (24) will actually happen. This is true even if there are obstacles in the channel, before the obstacles are reached. The fastest speed of internal waves is of the order of the square root of the density gradient if the density gradient is continuous, or of the square root of the density difference $\Delta\rho$ divided by the main density $\rho_m$ if there is a density discontinuity. If the density gradient and $\Delta\rho/\rho_m$ (if an interface is present) are all very small and the velocity $(U, V)$ determined from the free-surface drop not small, then the solution (24) is valid. If a stagnant layer is present, the solution given by (33) and (39) is valid under the same conditions. Obviously, if the free-surface drop is very small, internal waves can travel upstream, and the downstream stratification will have a far-reaching influence on the flow, which then cannot in general be described by (24), or by (33) and (39).

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